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A cell-centred arbitrary Lagrangian–Eulerian (ALE) method

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SUMMARY

We present an original and accurate unstructured cell-centred arbitrary Lagrangian–Eulerian algorithm devoted to the simulation of multi-material fluid flows. Copyright © 2007 John Wiley & Sons, Ltd.

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KEY WORDS: cell-centred scheme; Lagrangian scheme; ALE; hydrodynamics

1. INTRODUCTION

Arbitrary Lagrangian–Eulerian (ALE) methods were introduced by Hirt *et al.* [1] to retain only the advantages of the Lagrangian and the Eulerian frameworks, to optimize accuracy, robustness, and computational efficiency.

The goal of this paper is to describe an original cell-centred ALE strategy for multi-material fluid flows. The main elements in an ALE simulation are an explicit Lagrangian step, a rezoning step in which nodes of the Lagrangian grid are moved to improve the geometric quality of the grid, and a remapping step in which the Lagrangian solution is reconstructed on the rezoned grid. Classically, Lagrangian step uses a staggered scheme, in which velocities are vertex centred and the other variables are cell centred [2]. A difficulty with this approach lies in the fact that one needs special treatment for momentum remapping [3, 4]. As our Lagrangian step is fully cell centred [5], we avoid such special treatment. Moreover, we avoid need for hourglass suppression and artificial viscosity methods. Our rezoning step utilizes the 'local' minimization of nodally based objectives functions [6–9]. Furthermore, for multi-material fluid flows, we improve the quality of interface rezoning by repositioning it on a Bezier curve; thanks to the solution of a constraint minimization problem. The remapping step is based on an unstructured extension of the 'simplified face-based donor cell' method of [10]. The remainder of this paper is organized as follows. We first describe the different steps of the ALE formulation. Then, computational results are given to access the

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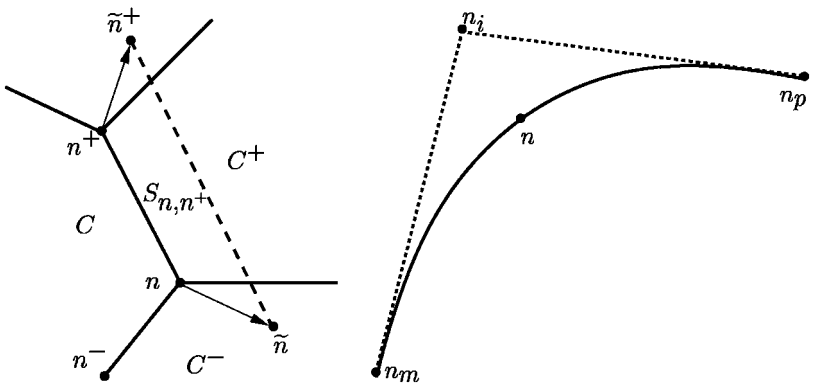

Figure 1. Notations in a cell (left) and Bezier's curve (right).

robustness and the accuracy of this method. Finally, we study the ablation of a plastic target due to laser irradiation.

2. NOTATIONS

Let $\{c\}$ be a collection of non-overlapping polygons whose union covers the domain filled by the fluid. The set of the polygons vertices is denoted by $\{n\}$. We denote the set of vertices of a particular cell c by $\mathcal{N}(c)$, and the set of cells that share a particular vertex n by $\mathcal{C}(n)$. For a node $n(x, y) \in \mathcal{N}(c)$, $n^-(x^-, y^-)$ and $n^+(x^+, y^+)$ are the previous/next nodes with respect to n in the list of vertices of cell c, see Figure 1 (left). Also, c^- and c^+ are the previous/next cells with respect to c in the list of cells c, see Figure 1 (left).

Then, we will denote all quantities related to the rezoned grid by the tilde accent.

3. LAGRANGIAN STEP

The Lagrangian step is based on a second-order cell-centred Lagrangian scheme [5]. The primary variables in this scheme are cell centred, i.e. specific volume, momentum, and total energy. The vertex velocities and the numerical fluxes through the cell interfaces are not computed independently contrary to standard approaches, but are evaluated in a consistent manner due to an original solver located at the nodes. This nodal solver can be viewed as a two-dimensional extension of the Godunov acoustic solver. The spatial second-order extension is derived using a MUSCL-type approach. Time discretization is based on a second-order Runge–Kutta scheme.

4. REZONING STEP

This section describes the method we use to improve grid quality. The rezoning problem consists of a 'local' minimization problem for the node-based objective functions that define the criteria for grid quality [6].

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Let $n(x, y)$ be a given internal node of the grid that is to be rezoned, and let the basic functional of our minimization procedure defined by

$$F(x, y) = \frac{1}{2} \sum_{c \in \mathcal{C}(n)} (1-\alpha) \underbrace{\frac{\|\mathbf{nn}^-\|^2 + \|\mathbf{nn}^+\|^2}{\det(J)}}_{F_S} + \alpha \underbrace{\frac{(\mathbf{nn}^- \cdot \mathbf{V}) + (\mathbf{nn}^+ \cdot \mathbf{V})}{\|\mathbf{V}\|^2}}_{F_V} + \underbrace{\frac{(\mathbf{nn}^- \cdot \mathbf{nn}^+)^2}{\det^2(J)}}_{F_O} \quad (1)$$

where $\mathbf{nn}^- = (x^- - x, y^- - y)$ and $\mathbf{nn}^+ = (x^+ - x, y^+ - y)$ are the edge vectors (see Figure 1(left)), $J = [\mathbf{nn}^-, \mathbf{nn}^+]$ is the associated Jacobian matrix, $\mathbf{V}$ is the fluid velocity, and $\alpha \in [0, 1[$ is a parameter.

Note that the rezoned position of $n(x, y)$ is determined by locally minimizing this local functional of two variables, with all other nodes held fixed at their Lagrangian positions. Furthermore, we take only a single step towards the minimum.

F_S is the 'smoothness' functional proposed by Knupp [6], which is closely related to the Winslow smoother [11]. In most of rezoning method, objective functions are based only on geometric considerations. However, the physical motion of the Lagrangian grid must be taken into account in the grid rezoning problem. One can constrain the rezoned grid to remain as close as possible to the Lagrangian grid. This is the philosophy of the so-called 'Reference Jacobian Method' developed by Knupp *et al.* [11]. In our rezoning strategy, we introduced an efficient directional control F_V, related to the fluid velocity [8]. The main advantage of this approach is the ease with which it is implemented. The parameter α can be viewed as a weighting control between geometric and physical criteria. It has to be defined by the user in the interval $[0, 1[$. Finally, because grid orthogonality is not controlled by the 'smoothness' functional, we use the specific functional F_O [9].

For multi-material fluid flows that include interface, we resolve a constraints minimization problem to improve the quality of the interface rezoning: min $F(x(t), y(t))$, where $x(t)$ and $y(t)$ are defined by the Bezier's curve:

$$\begin{aligned} x(t) &= (1-t)^2 x_m + 2(1-t)t x_i + t^2 x_p \\ y(t) &= (1-t)^2 y_m + 2(1-t)t y_i + t^2 y_p \end{aligned} \quad (2)$$

where n_m and n_p are the previous/next nodes with respect to n on the interface (see Figure 1 (right)) and $t \in [0, 1]$. Moreover, n_i is such that n is on the curve:

$$x_i = \frac{x - (1-t_0)^2 x_m - t_0^2 x_p}{2(1-t_0)t_0}, \quad y_i = \frac{y - (1-t_0)^2 y_m - t_0^2 y_p}{2(1-t_0)t_0} \quad (3)$$

where t_0 is set to 0.5.

Interface can be a boundary or an internal interface between different materials. Furthermore, during the remapping step, we set the numerical fluxes to zero through the interfaces in order to obtain a quasi-Lagrangian interface tracking.

The next step of the procedure performs a global control and an improvement in the geometric quality of the grid [7], when previous procedures cause the grid to become tangled or non-convex. The need of such a procedure also exits when the Lagrangian step creates non-valid elements in a grid.

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5. REMAPPING STEP

The remapping step is an interpolation procedure of mass, momentum, and total energy, from the Lagrangian grid to the rezoned one. In our context, we consider that the Lagrangian step grid and the rezoned one have the same connectivity.

For each displacement of two consecutive vertices n and n^+, we can build a quadrangle $S_{n,n^+} = \{n, \widetilde{n}, \widetilde{n}^+, n^+\}$, see Figure 1 (left). The signed volume of a rezoned cell is computed as the sum of the old cell volume and all the swept volume $V(S_{n,n^+})$ by the faces of c: $V(\widetilde{c}) = V(c) + \sum_{\{n,n^+\}} V(S_{n,n^+})$.

Let $\psi \equiv \psi(x, y)$ be a positive density function of the flow, $\psi = \rho, \rho u, \rho v, \rho E$. Let $m(c)$ be the mean mass value of the fluid enclosed in c, associated with the mean density value $\psi(c)$. The remapping problem is to compute in the rezoned cell $\widetilde{c}$, an accurate approximation of $\widetilde{m}$, in which several conditions must be satisfied to guarantee accuracy, conservativity, linearity preservation, and local-bound preservation [10].

The mass of the rezoned cell can be written as the mass of the corresponding old cell plus the exchange of masses with the neighbouring cells:

$$\widetilde{m} = m + \sum_{\{n,n^+\}} \Psi_{n,n^+} \tag{4}$$

where Ψ_{n,n^+} represents the flux exchanged through the face $\{n, n^+\}$ between the cell c and all the cells swept by the displacement of this face. Here, the approximate fluxes are

$$\Psi_{n,n^+} = \int_{S_{n,n^+}} \bar{\psi}\, \mathrm{d}x\, \mathrm{d}y \quad \text{where } \bar{\psi} = \begin{cases} \psi(c^+) & \text{if } V(S_{n,n^+}) \geqslant 0 \\ \psi(c) & \text{if } V(S_{n,n^+}) < 0 \end{cases} \tag{5}$$

Fluxes are computed using a piecewise linear reconstruction.

This cell-centred remapping is an unstructured extension of the 'simplified face-based donor-cell' method [10]. This approach does not need the computation of the intersections of the old grid and the corresponding rezoned one, which makes this approach much more efficient.

6. NUMERICAL RESULTS

All the numerical results have been obtained such that the rezoning/remapping is performed at each Lagrangian step, and α is set to 0.8.

6.1. Sedov test case

First, we present the propagation of a high-intensity cylindrical shock wave generated by a strong explosion. We present the numerical results with the data and the polygonal grid defined in [3]. The numerical solution preserves the cylindrical symmetry. One can see that in our ALE simulation, density profile is very close to the Lagrangian solution [3]. This is due to the introduction of a directional control of the grid rezoning, F_{V} in Equation (1) (Figures 2 and 3).

6.2. Laser beam ablation problem

In this problem we study the ablation of a plastic target (10^{-2} cm $\times$ 0.1 cm) due to laser irradiation. The peak intensity is set to 1.510^{15} W/cm^2. The initial density is set to 1.05 g/cm^3 and initial

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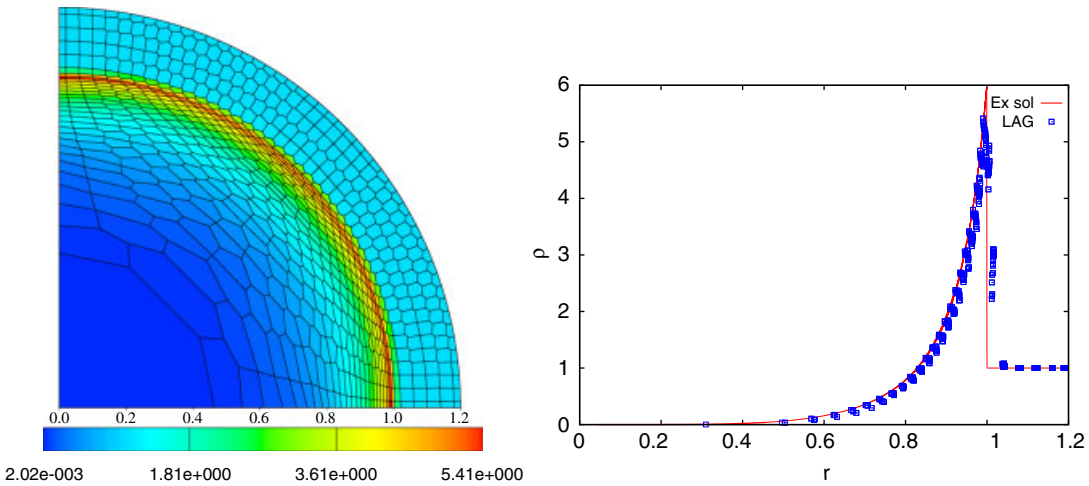

Figure 2. Sedov problem—Lagrangian solution. Mesh and density (left), and density profile in all the cells *versus* exact solution (right) at time $t = 1$.

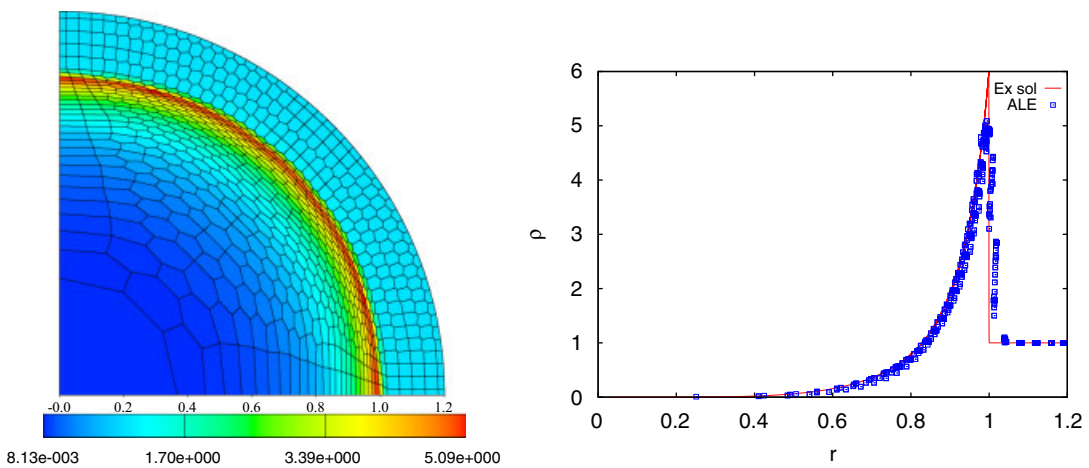

Figure 3. Sedov problem—ALE solution. Mesh and density (left), and density profile in all the cells *versus* exact solution (right) at time $t = 1$.

temperature is set to 369 K. Here, a perfect gas EOS is used for the plastic with $\gamma = \frac{5}{3}$ and non-linear thermal conduction is introduced. The initial grid is composed by 100×50 cells. The laser energy is released at the critical density. It produces a shock wave, followed by an ablation front. A plasma plume expands backside. We point out in Figure 4 that Bezier's curve improves the boundary calculations. Furthermore, as for Sedov results, by using the directional control functional F_V we keep the high grid density of the Lagrangian calculation in the shocked region.

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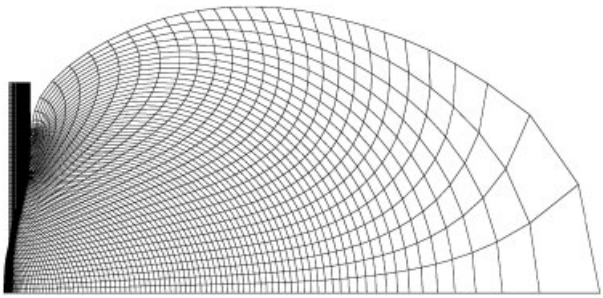

Figure 4. Laser beam ablation problem. Plasma plume at $t = 1.25$ ns.

7. CONCLUSION

We have shown that our original unstructured cell-centred ALE algorithm gives very accurate and efficient results on academic problems. Furthermore, our method improves the robustness of ALE calculations on plasma physics problems. In the future we plan to incorporate interface reconstruction in order to obtain a truly Lagrangian interface tracking.

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